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String fluid, tachyon matter, and domain walls

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Abstract: We study classical dynamics of an open string tachyon T of unstable Dp-brane coupled to the gauge field A_μ. In the vacuum with vanishing potential, $V=0$, two fluid-like degrees of freedom, string fluid and tachyon matter, survive the tachyon condensation. We offer general analysis of the associated hamiltonian dynamics in arbitrary background. The canonical field equations are organized into two sets, fluid equations of motion augmented by an integrability condition. We show that a large class of motionless and degenerate family of classical solutions exist and represent arbitrary transverse distribution of tachyon matter and flux lines. We further test their stability by perturbing the fluid equation up to the second order.

Second half of this note considers possibility of $V\neq 0$ in the dynamics. We incorporate V in the hamiltonian equation of motion and consider interaction between domain walls and string fluid. During initial phase of tachyon condensation, topological defect at $T=0$ is shown to attract nearby and parallel flux lines. The final state is fundamental strings absorbed and spread in some singular D$(p-1)$ brane soliton. When string fluid is transverse to the domain wall, the latter is known to turn into a smooth solution. We point out that a minimal solution of this sort exists and saturates a BPS energy bound of fundamental string ending on a D$(p-1)$ brane.

Keywords: Tachyon Condensation, D-branes.

Contents

1. String fluid and tachyon

Decay of unstable D-branes [1] have served as useful laboratories of understanding off-shell structure of string theory. After several years of intense study, we seem to have fairly good idea how lower dimensional, stable D-branes are formed out of decaying unstable D-branes. They arises as topological defects, familiar in ordinary field theories, although solutions tends to be a bit singular than usual [2, 3, 4]. Among very well understood are how the relevant space-time Ramond-Ramond charges are generated via winding numbers and topology of gauge bundles [5].

One of reasons that we should anticipate lower dimensional D-branes to emerge at the end of day is the charge conservation. Stable D-branes carry Ramond-Ramond gauge charge [6], which can be absorbed in unstable D-brane and transmutes into world-volume gauge field configurations [7]. Charge conservation in the space-time begs for a mechanism for recovering these conserved quantum numbers, and the only objects that can carry such charge after the unstable brane annihilate itself, would be the stable D-branes themselves. Thus, we must somehow be able to reproduce the D-branes within the dynamics of unstable D-branes.

One would expect the same logic should apply to fundamental strings [8, 9], say, to infinitely long semiclassical ones at least. Yet, emergence of fundamental strings has proved a much more challenging problem. Some tantalizing hints have been accumulating via study of low energy effective action of D-brane decay, nevertheless, and in this note, we will give a comprehensive review of the classical low energy dynamics with a commonly employed effective action, conceived by many authors, and study various aspects with a view toward formation of fundamental string. The form of the lagrangian we will study is of the form [10]–[16],

$$-V(T)\sqrt{-\,\mathrm{Det}(\eta + F + \nabla X^I \nabla X^I + \nabla T \nabla T)}\,, \tag{1.1}$$

where the tachyon T appears inside the determinant on equal footing as the transverse scalars X^I. $V(T)$ is everywhere nonnegative and has a runaway behavior [17, 18, 19]. That is $V(T)$ vanishes exponentially at $T = \pm\infty$. For first half of this note, we will consider $V = 0$ strictly. Latter part of the note will incorporate $V \neq 0$ and compute its effect near domain wall formation.

The earliest evidence that this low energy field theory is quite unconventional came in ref. [9], which considered pure gauge dynamics in decay of unstable D2-brane. They found that the electric flux lines become free in that no transverse pressure is present and also that the tension density of the flux lines obeys a BPS-like property which would be normally seen in fundamental string. This degenerate behavior of flux lines were later found to persist in higher dimensional case as well [20], once $V(T)$ is taken to vanish.

This pressureless collection of flux lines were dubbed "string fluid," in obvious reference to the fact that they carry fundamental string charges. This fluid-like behavior had been also noticed previously, in the context of a strong coupling limit of Born-Infeld system [21, 22, 23]. One outstanding question is whether and how properly quantized fundamental string emerges from this string fluid. Tantalizingly similarities already exist between string fluid and classical fundamental string raising hope that via some confinement mechanism we may recover fundamental string from decay of unstable D-branes. Classical properties of string fluid are studied in detail in ref. [20].

It turns out such fluid-like behavior is not limited to the gauge sector but extends to the tachyon. When the final state is static and homogeneous, an on-shell condition reads [20, 24, 25]

$$1 = E_i E_i + \dot{T}^2 \,, \tag{1.2}$$

where $E_i = F_{0i}$ are components of the field strength. Energy density of any motionless state is composed of two components

$$H_{motionless} = \sqrt{\pi^i \pi^i + \pi_T^2} \,, \tag{1.3}$$

where π^i is the conserved electric flux, such that $\nabla_i \pi^i = 0$, while π_T is the canonical momenta of T. The kinematics are such that we have a relationship,

$$\frac{\pi_i}{\pi_T} = \frac{E_i}{\dot{T}} \,. \tag{1.4}$$

Pure string fluid emerges when $\dot{T} = 0 = \pi_T$, while the other limit $\vec{E} = 0$ involves energy density composed solely of π_T. In the latter limit, all energy is carried by dust-like matter, known as tachyon matter [26, 27].

One of less understood aspects of the combined system of string fluid and the tachyon matter is how the two components interact with each other [25, 28, 29, 37] This system was addressed comprehensively and without any approximation, in ref. [25]. Among explained are the origin of the pressureless nature of these fluid, and also exact hamiltonian, canonical field equations, and energy-momentum tensor are written down. The combined fluid shows character of 1+1 dimensional system with variable "speed of light", which depends on the composition of the two fluid. Although exact "static" solutions were found, generic aspect of the dynamics has been poorly understood. In this note, we will address this issue and try to understand how one fluid component react to the presence of the other.

We provide a comprehensive analysis of the classical dynamics in the hamiltonian formulation, taking into account possible coupling to background metric. One purpose is to clarify the relation between the fluid equation of motions and the hamiltonian equation of motion. Previously, the former has been derived from combination of the latter and the energy-momentum conservation. In this note, we will describe in what sense solution to the fluid equation gives solutions to the field equations. Sections 2, 3, and 4 are devoted to this.

In particular, this allows us a rather sweeping characterization of all static (meaning that physical momenta vanishes identically) classical solutions and shows their rather huge degeneracy. After isolating such static solutions, we test their stability by perturbing the fluid system in section 5. Perturbation up to 2nd order is performed, which looks pretty involved, and we argue that all physical effect from such perturbation may be understood as a consequence of continuity of the fluid and lacks any destabilizing interaction.

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In fact, the aggregation of energy density and flux lines turns out to be tied to formation of domain walls, instead. In section 6, we extend the hamiltonian dynamics formulation to include possibility $V \neq 0$. We show that the domain-wall tends to attract nearby and parallel flux lines via a short range attraction. The effective range of this attraction is given by region with finite V, and collapses as stable D$(p-1)$ branes form. However, once the domain wall formation is complete, the range of attractive interaction become arbitrarily small, and such aggregation of (parallel) string fluid is no longer favored.[1]

In section 7, we consider another configuration involving string fluid and domain wall, where the former lies orthogonal to the latter. Smooth solutions of this kind were recently written down, and here we reproduce them from our hamiltonian description above. In particular, we show that a minimal solution exists and saturate BPS bound which is normally associated with 1/4 BPS configuration of fundamental string(s) ending on or passing through a D-brane. We close with a summary.

2. Fluid equation, integrability condition, and classical solutions

Fluid equation for $V = 0$ was first written down in ref. [20]. The main purpose there was to understand gauge dynamics, but the analysis did deal with possibility of turning on transverse scalars and also dynamical T provided that the kinetic term of the latter shows up on equal footing as transverse scalars X^I. This is precisely the lagrangian in question. Furthermore, any scalar whose kinetic term appears this way can be treated as if it is a component of gauge field along some hidden direction, and manipulations for gauge field carries over almost verbatim. Because of this, the result carries over immediately to the system of string fluid coupled to tachyon matter. In this section, we will rewrite the result with tachyon field explicitly expressed, and consider its implications. In section 3 and 4, we will elevate this to the general background with curved metric.

The main idea is to deal with the dynamics from the hamiltonian viewpoint. Let us introduce extended gauge field $A_M = (A_\mu, A_T)$ so that $A_T \equiv T$. Then the lagrangian of tachyon coupled to a gauge field may be written succinctly as [30]

$$-V(T)\sqrt{-\det(\eta_{MN}+F_{MN})}\,, \tag{2.1}$$

where F_{MN} is the field strength associate with A_M with the understanding that $\partial_T \equiv 0$. Since we will be using hamiltonian formulation we will be separating out time direction from the "spatial" ones. For the latter we will use subscript $m = 1, \ldots, p, T$ and $i = 1, \ldots, p$. Using Legendre transformation, we obtain the following hamiltonian of this system

$$\mathcal{H} = H - A_0 \partial_i \pi^i\,, \tag{2.2}$$

where

$$H = \sqrt{\pi^m\pi^m + P_m P_m + V^2 \det(\delta_{mn}+F_{mn})}\,. \tag{2.3}$$

[1] We must caution readers that this effect is unrelated to usual attraction between D-branes and parallel fundamental string. The latter arises from exchange of closed strings between the two objects, and thus is normally associated with open string one-loop. The net effect is the same, nevertheless, so such a tendency to form bound state of a D$(p-1)$ and fundamental strings is not too surprising.

Here π^m is the canonical conjugate momenta, defined as $\partial\mathcal{L}/\partial\dot{A}_m$, while $P_m \equiv \pi^n F_{mn}$. P_i with $i = 1, 2, \ldots, p$ correspond to the conserved momenta associated with the translational symmetry.

In section 2 through section 5, we will consider the limit, $V = 0$, which represent the final stage of tachyon condensation where tachyon T is rolling away to ∞ everywhere. The hamiltonian (in temporal gauge) is exceedingly simple, and has

$$H = \sqrt{\pi^m\pi^m + P_m P_m}\,. \tag{2.4}$$

Note that despite the vanishing lagrangian as $V \to 0$, the hamiltonian remains finite.

2.1 Fluid equations

Half of the canonical equation of motion

$$\dot{\pi}^m = -\frac{\delta}{\delta A_m}\int dx^p\,\mathcal{H}\,, \tag{2.5}$$

combined with conservation of energy momentum gives immediately the following fluid equations [20]

$$\begin{aligned} \partial_0 n^m + v^i\partial_i n^m &= n^i\partial_i v^m\,, \\ \partial_0 v^m + v^i\partial_i v^m &= n^i\partial_i n^m\,, \end{aligned} \tag{2.6}$$

where the vector fields n and v are defined as

$$\begin{aligned} \pi &= Hn\,, \\ P &= Hv\,, \end{aligned} \tag{2.7}$$

which satisfies the constraints

$$n^m n^m + v^m v^m = 1\,, \qquad n^m v^m = 0\,. \tag{2.8}$$

The evolution of the energy density H is then determined via,

$$\partial_0 H + \partial_i\left(Hv^i\right) = 0\,, \tag{2.9}$$

and finally π^i must satisfy Gauss's constraint,

$$\partial_i\pi^i = 0\,. \tag{2.10}$$

The Noether momenta P_i has the following simple expression

$$P_i = -F_{ij}\pi^j - \partial_i T\pi_T = \pi_m F_{mi}\,. \tag{2.11}$$

The last component of the vector P, P_T, is not really a conserved momenta since T-th direction is a mere mathematical device of convenience. It is nevertheless computed by pretending T-th direction exists as a translationally invariant spatial direction, and thus given by the combination

$$P_T \equiv \partial_i T\pi^i = \pi_m F_{mT}\,. \tag{2.12}$$

Most of quantities here have simple physical interpretation. π^i is nothing but conserved electric flux, while π_T is the tachyon matter density. H and P_i are conserved energy and momentum density, respectively. The last quantity P_T measure inhomogeneity of tachyon T along the flux direction. The fact that this appears in the hamiltonian separately implies a rather anisotropic behavior the system.

2.2 Integrability conditions

Although fluid equations are self-contained, a solution may not give automatically a solution to the original field equations. This is because that the fluid variables above are naturally formed from canonical variables and does not produce elementary fields A_m directly. A further set of first-order equations must be solved. These arise from

$$\dot{A}_n = \frac{\delta}{\delta\pi^n}\int dx^p\, \mathcal{H}\,, \tag{2.13}$$

and are exactly half of the hamiltonian equations of motion.

These equations are intimately related to the fact that the determinant part of the lagrangian vanishes. The hamiltonian with V kept is such that

$$H = \sqrt{\pi_m C_{mn}\pi_n + O(V^2)}\,, \tag{2.14}$$

for some matrix C independent of π_m. Then the lagrangian is

$$L = \dot{A}_m\pi_m - \mathcal{H} = \pi_m\left(\frac{\delta}{\delta\pi^m}\int\mathcal{H}\right) - \mathcal{H} = O(V^2)\,. \tag{2.15}$$

On the other hand the lagrangian is of the form

$$-V\sqrt{-\det(\cdots)}\,, \tag{2.16}$$

so the on-shell value of lagrangian must be such that

$$\sqrt{-\det(\cdots)} \sim V \to 0\,, \tag{2.17}$$

which is precisely the condition that generalizes,

$$1 - \dot{T}^2 - E_iE_i \to 0\,, \tag{2.18}$$

of the homogeneously rolling tachyon.

In any case, these equations can be expressed as an algebraic equation for the field strength associated with elementary fields A_q in terms of fluid variables, H, n, v;

$$E_m = n_m + F_{mn}v_n\,, \tag{2.19}$$

where $E_m = F_{0p}$. This, combined with the Bianchi identity

$$\dot{F}_{mn} = \partial_m E_n - \partial_n E_m\,, \tag{2.20}$$

gives an evolution equation for F_{mn}.

On the surface, thus, it may seem that we have split the system into two steps; solve the first-order fluid equation and then solve another first order equations for elementary fields in the background of H, n, and v. However, there is one subtlety here. The two fluid variables v and n are algebraically related as

$$v_m = n_n F_{nm} \,, \tag{2.21}$$

and there is a logical possibility that one may not find such F_{mn} as a solution. Only if there is a solution F_{mn} consistent with (2.21), the solution to fluid equation would be acceptable.

In the simplest case of pure tachyon, this integrability condition appears in a particularly simple manner. The only elementary field is $A_T = T$ in that case, and we have $n_i = 0$. Then (2.21) states that

$$v_i = -\partial_i T n_T = \mp \frac{\partial_i T}{\sqrt{1 + (\partial_i T)^2}} \,, \tag{2.22}$$

while (2.19) gives,

$$\dot{T} = \left(1 + (\partial_i T)^2\right) n_T = \pm\sqrt{1 + (\partial_i T)^2} \,. \tag{2.23}$$

In this case the integrability condition may be imposed as a familiar local condition on the fluid system as

$$\partial_i \left(\frac{v_j}{n_T}\right) - \partial_j \left(\frac{v_i}{n_T}\right) = 0 \,. \tag{2.24}$$

In section 3, we will further consider the pure tachyon case and reformulate these conditions in a manifestly relativistic manner.

2.3 "Static" solutions

Let us characterize all static solutions. By "static" we mean absence of physical momentum, $P_i = 0$, and this immediately constrains us to look for configurations with $v_i = 0$. One of the algebraic constraint then implies that $n^T v_T = 0$. This can be achieved by $v_T = 0$ in addition to $v_i = 0$, and if we choose this,[2] fluid equation collapse to

$$\partial_0 H = 0 \,, \tag{2.25}$$
$$\partial_0 n^m = 0 \,, \tag{2.26}$$

and

$$n^i \partial_i n^m = 0 \,, \tag{2.27}$$
$$\partial_i \left(H n^i\right) = 0 \,. \tag{2.28}$$

Because of (2.27), flux lines should be straight, and we may as well associate its direction with x^1 while denoting the remaining $p-1$ directions by $x^{2,3,\ldots,p}$. The most general "static"

[2]The other choice of $n_T = 0$ leads to a different class of solution that involves domain walls [31, 32, 33]. Since $0 \neq v_T = n_i \partial_i T$ implies nonvanishing gradient of T along the flux direction, such a configuration will gave $T = 0$ somewhere along the flux line direction. The final state would corresponds to a domain wall threaded by a transverse string fluid. Because of this, one can no longer restrict to $V \to 0$ limit, and one must solve for the full equation of motion. See section 7 for this class of solutions.

solutions (with $v_T = 0$) to the fluid equations are time-independent and x^1-independent distribution of flux π^1 and tachyon matter π^T. By construction the same property holds for H. Then, all "static" classical solutions are characterized by the following two arbitrary and independent (nonnegative) functions[3]

$$\pi^1(x^2, x^3, \ldots, x^p)\,, \tag{2.29}$$

$$\pi^T(x^2, x^3, \ldots, x^p)\,, \tag{2.30}$$

together with the choice of x^1 direction or equivalently the choice of the electric flux direction. The energy density is

$$H(x^2, \ldots, x^p) = \sqrt{(\pi^1(x^2, x^3, \ldots, x^p))^2 + (\pi^T(x^2, x^3, \ldots, x^p))^2}\,. \tag{2.31}$$

In section 5, we will analyze classical stability of this large family of static solutions.

We still need to test whether this "static" solution satisfies all the integrability condition. Namely, we need to find a gauge field, A_m, solving (2.19), such that (2.21) holds. Since $\vec{v} = 0$ by construction, we have

$$E_m = n_m = \frac{\pi_m}{H}\,, \tag{2.32}$$

which has a solution in the temporal gauge $A_0 = 0$

$$A_m = t n_m(x^2, \ldots, x^p)\,. \tag{2.33}$$

Then the magnetic field is

$$\begin{aligned} -F_{1j} &= F_{j\,1} = t\partial_j n_1\,, \\ -F_{Tj} &= F_{jT} = t\partial_j n_T\,, \end{aligned}$$

for $j = 2, 3, \ldots, p$. All other components vanish. Testing whether this gives back a vanishing v via $v_m = n_n F_{nm}$, we find

$$\begin{aligned} v_1 &= 0\,, \\ v_j &= -F_{j\,1}n_1 - F_{j\,T}n_T = -t\partial_j\left(\frac{n_1^2 + n_T^2}{2}\right) = 0\,, \\ v_T &= 0\,. \end{aligned} \tag{2.34}$$

Thus our "static" solution is integrable and thus acceptable as a solution to the original field equations.

3. Tachyon matter in general background

The tachyon effective action on an unstable Dp brane in general gravity background is given by

$$S = -\int d^{p+1}x V(T)\sqrt{-g}\sqrt{1 + g^{\mu\nu}\partial_\mu T\partial_\nu T}\,, \tag{3.1}$$

[3] An exceptional case is when π^i also happens to vanish; π^T could then be an arbitrary function of $x^{1,2,3,\ldots,p}$.

where $g \equiv \det g_{\mu\nu}$ and $V(T)$ is the tachyon potential, and we defined the general metric as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + L^i dt)(dx^j + L^j dt) \\ &= g^{\mu\nu}\partial_\mu \partial_\nu = -\frac{1}{N^2}(\partial_t - L^i \partial_i)^2 + h^{ij}\partial_i \partial_j \,, \quad (1 \le \; i,j \; \le p)\,. \end{aligned} \tag{3.2}$$

This general metric is determined by the lapse function N, shift vector L^i, and spatial metric h_{ij} of the p-dimensional hyper-surface at a given time slice. Using this general metric (3.2), we can rewrite the action (3.1) as

$$\begin{aligned} S &= \int dt \int d^p x \, \mathcal{L}\,, \\ \mathcal{L} &= -V(T) N \sqrt{h}\sqrt{\tilde{X}}\,, \end{aligned} \tag{3.3}$$

with $h \equiv \det h_{ij}$, $g = -N^2 h$ and

$$\begin{aligned} \tilde{X} &\equiv 1 + g^{\mu\nu}\partial_\mu T \partial_\nu T \\ &= 1 - \frac{1}{N^2}(\dot{T} - L^i \partial_i T)^2 + h^{ij}\partial_i T \partial_j T\,, \end{aligned} \tag{3.4}$$

where the doted notation denotes time derivative.[4] Then the hamiltonian is obtained by

$$\begin{aligned} \mathcal{H} &= \pi \dot{T} - \mathcal{L} \\ &= \pi \dot{T} + V(T) N \sqrt{h}\sqrt{\tilde{X}} \\ &= N\sqrt{\pi^2 + (\pi\partial_i T) h^{ij} (\pi \partial_i T) + V^2 h (1 + h^{ij}\partial_i T \partial_j T)} + \pi L^i \partial_i T\,, \end{aligned} \tag{3.5}$$

in which we have defined the conjugate momentum as

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{T}} = \frac{\sqrt{h} V}{N\sqrt{\tilde{X}}}(\dot{T} - L^i \partial_i T)\,. \tag{3.6}$$

After tachyon condensation, i.e., in $V \to 0$ limit, the hamiltonian equations are given by

$$\dot{T} = \frac{\partial \mathcal{H}}{\partial \pi} = N\sqrt{1 + h^{ij}\partial_i T \partial_j T} + L^i \partial_i T\,, \tag{3.7}$$

$$\dot{\pi} = -\frac{\partial \mathcal{H}}{\partial T} = \partial_i \left(\frac{N \pi h^{ij} \partial_j T}{\sqrt{1 + h^{kl}\partial_k T \partial_l T}} \right) + \partial_i (\pi L^i)\,. \tag{3.8}$$

Let us consider the following Lorentz invariant matter density

$$\begin{aligned} \mu \equiv \frac{p^\alpha \partial_\alpha T}{\sqrt{-g}} &= \frac{V}{\sqrt{\tilde{X}}}\left(\frac{1}{N^2}(\dot{T} - L^i \partial_i T)^2 - h^{ij}\partial_i T \partial_j T \right) \\ &= \frac{V}{\sqrt{\tilde{X}}}\,, \end{aligned} \tag{3.9}$$

[4] Ref. [35] also has some discussion on formulation of Born-Infled theory with vanishing tension in general background, with emphasis on constraint algebra.

where we define

$$p^\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha T)}\,, \tag{3.10}$$

so that $p^0 = \pi_T$. In the last step of the above equation, we used the eq. (3.7).

Using the equations (3.7), (3.8), we obtain

$$\begin{aligned}
\nabla_\mu(\mu\nabla^\mu T) &= \frac{1}{\sqrt{-g}}\partial_\mu(\mu\sqrt{-g}g^{\mu\nu}\partial_\nu T) \\
&= \frac{1}{\sqrt{-g}}\partial_0\left[\mu\sqrt{-g}\left(-\frac{1}{N^2}\,\dot{T} + \frac{L^i}{N^2}\partial_i T\right)\right] \\
&\quad + \frac{1}{\sqrt{-g}}\partial_i\left[\mu\sqrt{-g}\left(\frac{L^i}{N^2}\dot{T} + h^{ij}\partial_j T - \frac{L^i L^j}{N^2}\partial_j T\right)\right] \\
&= \frac{1}{\sqrt{-g}}\left[-\partial_0\pi + \partial_i(\pi L^i) + \partial_i\left(\frac{N\sqrt{h}V}{\sqrt{\tilde{X}}}h^{ij}\partial_j T\right)\right] = 0\,.
\end{aligned} \tag{3.11}$$

We used the eq. (3.8) in the last step of the eq. (3.11). Then the energy momentum conservation implies that

$$\begin{aligned}
0 = \nabla^\mu T_{\mu\nu} &= \nabla^\mu\left(\frac{V}{\sqrt{\tilde{X}}}\partial_\mu T\partial_\nu T\right) \\
&= \mu\partial_\mu T\nabla^\mu\partial_\nu T\,.
\end{aligned} \tag{3.12}$$

All of these have a rather obvious interpretation once we identify μ as the invariant matter density and $-\partial_\mu T$ as the velocity field U_μ [34]. Equations (3.11) and (3.12) then implies,

$$\nabla_\alpha\left(\mu U^\alpha\right) = 0\,, \tag{3.13}$$

$$U^\alpha\nabla_\alpha U^\mu = 0\,, \tag{3.14}$$

which are nothing but the continuity equation and the geodesic equation for dust. These are the two fluid equation for case of pure tachyon.

If we were treating the density μ and velocity field U_μ as elementary quantities, the fluid equations (3.14) must be augmented by an integrability condition

$$\nabla_\mu U_\nu - \nabla_\nu U_\mu = 0\,, \tag{3.15}$$

to make contact with the original field theory. This of course gives $U_\mu = -\partial_\mu T$ for some function T, and we would interpret this T as the original tachyon of the system. Also recoverable from the field equations is the fact that

$$U_\mu U^\mu = -1\,, \tag{3.16}$$

which is a kinematical constraint, saying that the velocity field $U_\mu = -\partial_\mu T$ arises from affine parameterization of the geodesics. Obviously it does not affect motion of the tachyon matter, and simply gives how T should be solved for, given any particular set of trajectories of tachyon matter.

Thus the tachyon matter in the tachyon condensation limit corresponds with an ideal (rotationless) fluid, moving freely along geodesics, with the trajectory being affine-parameterized. In particular, this implies that the tachyon matter clusters under the gravitational interactions just as ordinary matter does.

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4. String fluid coupled to tachyon matter

Here we repeat the above analysis with gauge field included. The fluid equation for string fluid were first written in ref. [20], and here we show their completeness again in that all nontrivial dynamical information can be recovered just from solving the fluid equations. The effective action for tachyon coupled to abelian gauge field on Dp-brane can be written by

$$S = \int d^{p+1}x\mathcal{L}\,, \qquad (4.1)$$
$$\mathcal{L} = -V(T)\sqrt{-X}$$

with

$$X \equiv \det(g_{\mu\nu} + \partial_\mu T\partial_\nu T + F_{\mu\nu})\,, \qquad (\mu, \nu = 0, 1, \ldots, p)\,, \qquad (4.2)$$
$$F = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

where $g_{\mu\nu}$ is defined in eq. (3.2).

In calculating the determinant in eq. (4.2), it is convenient to consider the tachyon field T as $(p + 1)$-component of the gauge field, as we already used in section 2. The metric is also extended to include a flat fictitious direction x^T. Then the eq. (4.2) can be rewritten by

$$X = \det(g_{MN} + F_{MN})\,, \qquad (M, N = 0, 1, \ldots, p, T)\,, \qquad (4.3)$$

where $F_{\mu T} = \partial_\mu T$ since $\partial_T = 0$. Using Legendre transformation, we obtain the following hamiltonian of this system

$$\mathcal{H} = N\sqrt{\pi^m h_{mn}\pi^n + P_m h^{mn} P_n + V^2 \det(X_{mn})} - P_m L^m - A_0\partial_i\pi^i\,, \qquad (4.4)$$

where $\pi^m \equiv \delta\mathcal{L}/\delta\dot{A}_m$, $P_m \equiv \pi^n F_{nm}$. Canonical field equations for this system can be found in ref. [25], and in the following we will reexpress these in terms of fluid-like variables and also generalize it to include nontrivial background metric.

4.1 Generalities: case of an unstable D2-brane

Our purpose in this paper is to understand the interaction between tachyon matter and electric flux line. To accomplish this purpose, we concentrate on the simplest case, D2-brane. We believe that the tachyon condensation on D2-brane contains all nontrivial characteristics of the interaction of tachyon and flux in general Dp-brane decay. The determinant in eq. (4.2) on D2-brane system is given by

$$X = g\left(1 + \frac{1}{2}F^2 - \frac{1}{16}(F^*F)^2\right) = g\tilde{X}\,, \qquad (4.5)$$

with

$$F^2 = F_{MN}F^{MN}\,, \qquad F^*F = F^*{}_{MN}F^{MN}\,, \qquad F^{*MN} = \frac{1}{2\sqrt{-g}}\epsilon^{MNPQ}F_{PQ}\,, \qquad (4.6)$$

where $M,N = 0,1,2,T$ and ϵ^{MNPQ} is the Levi-Civita symbol and we choose $\epsilon_{012T} = 1$, $\epsilon^{012T} = -1$. Energy-momentum tensor $T_{\mu\nu}$ is[5]

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{V}{\sqrt{\tilde{X}}}\frac{C^S_{\mu\nu}}{g}\,, \tag{4.7}$$

where C^S_{MN} is the symmetric part of the cofactor C_{MN} for $X_{MN} = g_{MN} + F_{MN}$ which is given by

$$C^{MN} = g\left(g^{MN}(1+\frac{1}{2}F^2) + F^{MN} + F^{MP}F_P{}^N - \frac{1}{4}F^{*MN}(F^*F)\right). \tag{4.8}$$

Equation of motion for the gauge field A_M and conservation of energy-momentum are expressed by

$$\nabla_\mu\left(\frac{V}{\sqrt{\tilde{X}}}\frac{C^{\mu N}_A}{g}\right) = 0\,, \tag{4.9}$$

$$\nabla_\mu\left(\frac{V}{\sqrt{\tilde{X}}}\frac{C^{\mu\nu}_S}{g}\right) = 0\,, \tag{4.10}$$

where C^{MN}_A is the anti-symmetric part of the cofactor in eq. (4.8). Conjugate momentum for gauge field and conserved Noether momentum which is induced by invariance of the spatial translation are given by

$$\pi^m = \frac{V}{\sqrt{\tilde{X}}}\frac{C^{0m}_A}{\sqrt{-g}}\,, \qquad P_i = \frac{V}{\sqrt{\tilde{X}}}\frac{C^{0m}_A}{\sqrt{-g}}F_{mi}\,. \tag{4.11}$$

In the next sections, we will do that in tachyon condensation limit in flat and curved space.

4.2 Fluid equations in flat space

In flat space metric $\eta_{MN} = \mathrm{diag}(-1,1,1,1)$, the hamiltonian density (2.2) in tachyon condensation limit ($V=0$) is given by

$$H = \sqrt{\vec{\pi}^2 + (\vec{B}\times\vec{\pi})^2} = \frac{V}{\sqrt{\tilde{X}}}(1+\vec{B}^2)\,, \tag{4.12}$$

where we use $A_0 = 0$ gauge and define $E_m \equiv F_{0m}$, $B_m \equiv \epsilon_{mnl}F_{nl}$ and thus

$$\tilde{X} = 1 + \vec{B}^2 - \vec{E}^2 - (\vec{E}\cdot\vec{B})^2 = 0\,, \tag{4.13}$$

with an explicit expression

$$\vec{E} = (E_1, E_2, \dot{T})\,, \qquad \vec{B} = (\partial_2 T, -\partial_1 T, B)\,. \tag{4.14}$$

With this, we find

$$\pi_m = \frac{V}{\sqrt{\tilde{X}}}\left(E_m + B_m(\vec{E}\cdot\vec{B})\right), \qquad P_m = (\vec{B}\times\vec{\pi})_m = \frac{V}{\sqrt{\tilde{X}}}\epsilon_{mnl}B_nE_l\,, \tag{4.15}$$

[5] Bear in mind that, since we artificially introduced x^T direction, there is no conserved momentum associated with this direction.

and under the condition (4.13),

$$\frac{V}{\sqrt{\tilde{X}}}C_A^{mn} = \frac{P_m\pi_n - \pi_m P_n}{H}\,,$$
$$\frac{V}{\sqrt{\tilde{X}}}C_S^{ij} = \frac{P_iP_j - \pi_i\pi_j}{H}\,. \tag{4.16}$$

Equations of motion gives

$$\vec{\partial}\cdot(H\vec{n}) = 0\,, \tag{4.17}$$
$$\dot{\vec{n}} + (\vec{v}\cdot\vec{\partial})\vec{n} = (\vec{n}\cdot\vec{\partial})\vec{v}\,, \tag{4.18}$$

when augmented with the energy conservation,

$$\dot{H} + \vec{\partial}\cdot(H\vec{v}) = 0\,. \tag{4.19}$$

The momentum conservation may be written as

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\partial})\vec{v} = (\vec{n}\cdot\vec{\partial})\vec{n}\,, \tag{4.20}$$

where energy conservation is taken into account. Actually, the last component of (4.20) does not arise from the conservation law, since no such conservation law exists for P_T. Rather it is a linearly dependent equation that may be derived from the rest. However, we include this time evolution equation for v_T for notational convenience.

As already described in section 2, the vectors $\vec{n}$ and $\vec{v}$ are defined as

$$\vec{n} \equiv \frac{\vec{\pi}}{H}\,, \qquad \vec{v} \equiv \frac{\vec{P}}{H}\,. \tag{4.21}$$

The two vectors, $\vec{n}$ and $\vec{v}$ satisfy the constraints

$$n^2 + v^2 = 1\,, \qquad \vec{n}\cdot\vec{v} = 0\,. \tag{4.22}$$

Thus, the dynamics produces a set of self-contained first-order fluid equations for H and $\vec{n}$ and $\vec{v}$. Although these equations are more natural in hamiltonian formulation, we stick to lagrangian formulation because the latter is more susceptible to incorporation of curved background.

4.3 Fluid equations in curved space

In the tachyon condensation limit ($V = 0$), the hamiltonian in curved space is written by

$$\mathcal{H} = N\sqrt{\pi^m h_{mn}\pi^n + P_m h^{mn} P_n} - L^m P_m\,, \tag{4.23}$$

where we use $A_0 = 0$ gauge and the X is defined in eq. (4.3) and to ensure the finiteness of hamiltonian density we have to set $X = 0$. After some calculation, we obtain the following two relations which is similar to the flat space case

$$\frac{V}{\sqrt{\tilde{X}}}C_A^{mn} = = \frac{P^m\pi^n - \pi^m P^n}{T^{00}}\,,$$
$$\frac{V}{\sqrt{\tilde{X}}}C_S^{mn} = = \frac{P^mP^n - \pi^m\pi^n}{T^{00}}\,, \tag{4.24}$$

where T^{00} is the (00)-component of the energy-momentum tensor defined in eq. (4.7), and we define the upper indexed P^i as

$$P^m \equiv g^{m0}\pi^n F_{n0} + g^{mn}P_n\,. \tag{4.25}$$

Using the relation in eq. (4.24), we can rewrite the equation of motion which is expressed in eqs. (4.9) as follows :

$$\begin{aligned} \nabla_i\left(\frac{\pi^i}{\sqrt{-g}}\right) &= 0\,, \quad (\nu = 0 \text{ case})\,, \\ \nabla_0\left(\frac{\pi^i}{\sqrt{-g}}\right) + \nabla_j\left(\frac{P^i\pi^j - P^j\pi^i}{\sqrt{-g}H}\right) &= 0\,, \quad (\nu = i \text{ case})\,, \end{aligned} \tag{4.26}$$

where

$$H \equiv \sqrt{-g}\,T^{00} = \frac{1}{N\sqrt{h}}\sqrt{\pi^m h_{mn}\pi^n + P_m h^{mn} P_n}\,. \tag{4.27}$$

And the conservation of energy-momentum (4.10) is given by

$$\begin{aligned} \nabla_0\left(\frac{H}{\sqrt{-g}}\right) + \nabla_i\frac{P^i}{\sqrt{-g}} &= 0 \quad (\nu = 0 \text{ case})\,, \\ \nabla_0\left(\frac{P^i}{\sqrt{-g}}\right) + \nabla_j\left(\frac{P^iP^j - \pi^i\pi^j}{\sqrt{-g}H}\right) &= 0 \quad (\nu = i \text{ case})\,. \end{aligned} \tag{4.28}$$

This is an analog of (4.17)–(4.20) in curved spacetime.

The form of fluid equations take particularly simple form when the shift vectors happen to vanish, $L^i = 0$ and N is constant. This would be the case when we are considering a cosmological scenario, for instance. With this, ∇_0 effectively collapses to $(1/N)\partial_t$, and the eqs. (4.26), (4.28) are summarized by

$$\vec{\nabla}' \cdot (H\vec{\tilde{n}}) = 0\,, \tag{4.29}$$

$$\dot{H} + \vec{\nabla}' \cdot (H\vec{\tilde{v}}) = 0\,, \tag{4.30}$$

$$\dot{\vec{\tilde{n}}} + (\vec{\tilde{v}}\cdot\vec{\nabla}')\vec{\tilde{n}} = (\vec{\tilde{n}}\cdot\vec{\nabla}')\vec{\tilde{v}}\,, \tag{4.31}$$

$$\dot{\vec{\tilde{v}}} + (\vec{\tilde{v}}\cdot\vec{\nabla}')\vec{\tilde{v}} = (\vec{\tilde{n}}\cdot\vec{\nabla}')\vec{\tilde{n}}\,, \tag{4.32}$$

where $\vec{\nabla}' = (\nabla'_1, \nabla'_2, 0)$ is the covariant derivative defined for the spatial part of the metric, h_{mn}, and the dot represent the time derivative $(1/N)\partial_t$. We have also defined

$$\tilde{n}^m \equiv \frac{\pi^m}{H}\,, \qquad \tilde{v}^m \equiv \frac{h^{mn}P_n}{H}\,, \tag{4.33}$$

with constraints

$$\tilde{n}^2 + \tilde{v}^2 = 1\,, \qquad \vec{\tilde{n}}\cdot\vec{\tilde{v}} = 0\,. \tag{4.34}$$

4.4 Integrability

Since the quantities that enter the definition of $\mathcal{H}$, $\vec{n}$, and $\vec{v}$, are naturally canonical variables, solving the first-order fluid equation will not generate the elementary fields T and A_μ directly. For these, we have to solve for another set of first order differential equations.

We could have derived the above fluid equation from the canonical formulation, and if we did so, half of the canonical equations would remain unused and simply relate conjugate momenta to time-derivative of elementary fields. At least when we make use of energy-momentum conservation directly. In terms of the above variables, the remaining equations may be written as

$$\vec{E} = \vec{n} - \vec{B} \times \vec{v} \,. \tag{4.35}$$

This determines the evolution of the "magnetic field" $\vec{B}$ via Bianchi identity as

$$\dot{\vec{B}} = \vec{\partial} \times \vec{n} - \vec{\partial} \times (\vec{B} \times \vec{v}) \,. \tag{4.36}$$

Thus, $\vec{B}$ may be solved for after $\vec{v}$ and $\vec{n}$ are determined. When this is done, $\vec{E}$ is also determined via (4.35).

However, since $\vec{B}$ enters the fluid degrees of freedom via the identity,

$$\vec{v} = \vec{B} \times \vec{n} \,, \tag{4.37}$$

we need to make sure that this does not further restrict solutions to the fluid equations alone. A priori, it is unclear whether for all solutions to the fluid equations we can find $\vec{B}$ such that it solves the evolution equation and is consistent with this algebraic constraint as well. Only if such a $\vec{B}$ exist for the solution to the fluid equation, we can say that the solution is physical.

In the curved background these integrability conditions are expressed as

$$E_m = \frac{1}{N\sqrt{h}} (h_{mn}\tilde{n}^n + F_{mn}\tilde{v}^n) - F_{mn}L^n \,, \tag{4.38}$$

$$h_{mn}\tilde{v}^n = F_{mn}\tilde{n}^n. \tag{4.39}$$

plus the Bianchi identity

$$\dot{F}_{mn} = \partial_m E_n - \partial_m E_n \,. \tag{4.40}$$

5. Stability analysis of classical solutions

In this section, we perform stability analysis of "static" solutions of section 2. Because of lack of transverse pressure, dimensionality of the unstable D-brane does not really matter, so we will confine our analysis to the case of unstable D2-brane. The authors of ref. [36] recently debated against the assertion that the degeneracy implies stability. Here we demonstrate the stability explicitly by solving dynamical equations of motion, and thereby invalidate the criticism.

Two cartesian coordinates are denoted as x and y. We will solve the eq. (4.17)–(4.20) in flat space using perturbation theory. The eqs. (4.18), (4.20) may be conveniently rewritten as

$$\dot{\vec{a}} - (\vec{b} \cdot \vec{\partial})\vec{a} = 0 \,, \qquad \dot{\vec{b}} + (\vec{a} \cdot \vec{\partial})\vec{b} = 0 \,, \tag{5.1}$$

where we define

$$\vec{a} \equiv \vec{n} + \vec{v} \,, \qquad \vec{b} \equiv \vec{n} - \vec{v} \,. \tag{5.2}$$

The constraints (4.22) become

$$a^2 = 1 \,, \qquad b^2 = 1 \,. \tag{5.3}$$

Let us then consider the following small fluctuations around the background fields $\vec{a}_0$, $\vec{b}_0$ which is given by the "static" classical solutions given in section 2. This means that $\vec{a}_0 = \vec{b}_0 = \vec{n}_0$ with

$$\vec{n}_0 = (n_{0x}(y), 0, n_{0T}(y)) \,. \tag{5.4}$$

Keep in mind here that $0 \leq (n_{0T})^2 = 1 - (n_{0x})^2 \leq 1$.

5.1 First order

Expanding around such a solution

$$\begin{aligned} \vec{a} &= \vec{a}_0 + \vec{a}^{(1)} + \vec{a}^{(2)}, \dots \,, \\ \vec{b} &= \vec{b}_0 + \vec{b}^{(1)} + \vec{b}^{(2)}, \dots \,, \end{aligned} \tag{5.5}$$

and using the eqs. (5.1), (5.5), we obtain the first order perturbation equations

$$\begin{aligned} \partial_- \vec{a}^{(1)} &= -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} b_y^{(1)} \,, \\ \partial_+ \vec{b}^{(1)} &= -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} a_y^{(1)} \,, \end{aligned} \tag{5.6}$$

where we define

$$x^\pm \equiv x \pm n_{0x} t \,, \qquad \partial_\pm = \pm \frac{1}{2n_{0x}} (\partial_t \pm n_{0x} \partial_x) \,.$$

The y-components of the first order perturbation equations in eq. (5.6) are expressed by two homogeneous first order differential equations

$$\partial_- a_y^{(1)} = 0 \,, \qquad \partial_+ b_y^{(1)} = 0 \,, \tag{5.7}$$

and the solutions are

$$a_y^{(1)} = f(x^+, y) \,, \qquad b_y^{(1)} = \tilde{f}(x^-, y) \,, \tag{5.8}$$

where $f(\tilde{f})$ is an arbitrary function with arguments $x^+(x^-)$ and y.

Substituting the solutions for $a_y^{(1)}$ and $b_y^{(1)}$ into the x-components of the eq. (5.6), we get

$$\partial_- a_x^{(1)} = -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} \tilde{f}(x^-, y) \,, \qquad \partial_+ b_x^{(1)} = -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} f(x^+, y) \,, \tag{5.9}$$

and the solutions are

$$a_x^{(1)} = g(x^+, y) - \frac{n'_{0x}}{2n_{0x}} \tilde{F}(x^-, y) \,, \qquad b_x^{(1)} = \tilde{g}(x^-, y) - \frac{n'_{0x}}{2n_{0x}} F(x^+, y) \,, \tag{5.10}$$

where g and $\tilde{g}$ are arbitrary functions, $n'_{0x} \equiv \partial_y n_{0x}$, and we define

$$F(x^+, y) \equiv \int^{x^+} dw \; f(w, y) \,, \qquad \tilde{F}(x^-, y) \equiv \int^{x^-} dw \; \tilde{f}(w, y) \,. \tag{5.11}$$

Using eq. (5.9), we obtain free wave equations, $\partial_+\partial_- a_x^{(1)}(b_x^{(1)}) = 0$, i.e.,

$$\left(\partial_t^2 - {n_{0x}}^2\partial_x^2\right) n_x^{(1)} = 0\,, \qquad \left(\partial_t^2 - {n_{0x}}^2\partial_x^2\right) v_x^{(1)} = 0\,, \tag{5.12}$$

this means that there are propagating modes along background flux line [25, 37] and no net effects which deforms the distribution of the background flux line and tachyon matter in the first other perturbation.

5.2 Second order: low frequency limit

Now let us consider the second order perturbation of the eq. (5.1). General formalisms for the second order perturbation are analyzed in appendix C. As a meaningful choice for the solution of the first order perturbation equations to investigate the second order ones, we consider the following configurations

$$f(x^+, y) = u(y) + c(x^+, y)\,, \tag{5.13}$$

$$\tilde{f}(x^-, y) = -u(y) + \tilde{c}(x^-, y)\,, \tag{5.14}$$

$$g(x^+, y) = -\frac{n'_{0x}}{2n_{0x}} u(y)x^+ + d(x^+, y)\,, \tag{5.15}$$

$$\tilde{g}(x^-, y) = \frac{n'_{0x}}{2n_{0x}} u(y)x^- + \tilde{d}(x^-, y)\,, \tag{5.16}$$

where $c(\tilde{c})$ and $d(\tilde{d})$ are arbitrary oscillatory functions. In other words, we allow a net velocity $u(y)$ along y-direction in the first order perturbation.

Since the second order perturbation is pretty involved, let us take some simplifying limit. We could for instance consider taking a very low frequency limit. After all, if there is a confining or dispersive effect, it should show up here. In the current general setup, we may achieve this by taking the limit

$$f_{osc}^{(m,n)}(x^\pm, y) \to 0\,, \tag{5.17}$$

where f_{osc} represent oscillatory piece and m, n are arbitrary positive integers. We use the notation,

$$A^{(m,n)}(x, y) \equiv \frac{\partial^{m+n}}{\partial x^m \partial y^n} A(x, y)\,. \tag{5.18}$$

In addition, we average over characteristic time-scale of the oscillatory pieces, meaning that we are mainly interested in net effect rather than exact time evolution of the system. We achieve this by averaging the equation over time where in effect we set

$$\langle f_{osc}\rangle \equiv \lim_{L\to\infty} \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dw\, f_{osc}(w, y) \to 0\,. \tag{5.19}$$

In this limit, the eqs. (C.12)–(C.15) are reduced to

$$(\partial_t^2 - {n_{0x}}^2\partial_x^2) n_x^{(2)} = 2uu'n'_{0x} + u^2 n''_{0x}\,, \tag{5.20}$$

$$(\partial_t^2 - {n_{0x}}^2\partial_x^2) v_x^{(2)} = 0\,, \tag{5.21}$$

$$(\partial_t^2 - {n_{0x}}^2\partial_x^2) n_y^{(2)} = 0\,, \tag{5.22}$$

$$(\partial_t^2 - {n_{0x}}^2\partial_x^2) v_y^{(2)} = 0\,. \tag{5.23}$$

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Left-hand-sides has the Klein-Gordon kinetic operator for an one-dimensional system with "speed of light" equal to $\sqrt{1-n_{0T}^2} = |n_{0x}|$. This is identical to the first order case, and implies that small fluctuation would move freely up and down along the flux lines, provided that there is no term on the right hand side.

On average, there is no net force on the second order fluctuation, generated from the first order fluctuation, except the right hand side of (5.20). However, this term represents a rather trivial effect. $u(y) \neq 0$ implies that the entire configuration is drifting along y direction. Since the flux lines and tachyon matter are distributed nontrivially along y direction, this motion will generate time-dependent change of $\vec{n}$. To see what effect it has, we need to solve for $v^{(2)}$ induced by $u(y)$ first. For x-independent perturbation, we have

$$\dot{\vec{v}} + (\vec{v}\cdot\vec{\partial})\vec{v} = 0\,, \tag{5.24}$$

which gives

$$v_y^{(1)}(y,t) + v_y^{(2)}(y,t) = u(y) - u(y)u'(y)t\,. \tag{5.25}$$

With this velocity field, we may ask how $\vec{n}(y,t)$ drifts with time. If the only physical effect is the drift, the $\vec{n}(y,t)$ will be identical to $\vec{n}(\tilde{y},0) = \vec{n}(y)$ where

$$y = \tilde{y} + \int_0^t ds\; v_y(f(s),s)\,, \tag{5.26}$$

with $f(s)$ is the y-trajectory between $f(0) = \tilde{y}$ and $f(t) = y$ due to the velocity field $v_y(y,t)$. Despite somewhat involved formulae so far, the relationship between y and $\tilde{y}$ is deceptively simple,

$$y = \tilde{y} + u(\tilde{y})t + O(t^3)\,, \tag{5.27}$$

or equivalently

$$\tilde{y} = y - u(y)t + u(y)u'(y)t^2 + O(t^3)\,. \tag{5.28}$$

Thus we find simple drift of the configuration along y-direction gives,

$$\begin{aligned}\vec{n}(y,t) &= \vec{n}(\tilde{y})\\ &= \vec{n}(y) - t\,(u(y)\partial_y\vec{n}(y)) + \frac{t^2}{2}\left(2u(y)u'(y)\partial_y\vec{n}(y) + u(y)^2\partial_y^2\vec{n}(y)\right),\end{aligned} \tag{5.29}$$

up to 2nd order in time t. Thus the nontrivial term on the right hand side of (5.20) simply represent this drift effect.[6]

5.3 Second order: case of interlocked distribution of the two fluid

One large subset of classical solutions we could consider in more detail is those with n_{0x} constant. Since n_{0x} is the ratio between the flux energy density and the total energy density, such a solution corresponds to an arbitrary distribution of the energy density

[6] We could have started perturbation after suppressing oscillatory pieces completely, instead of averaging over it, to see such drift effect. If we did that, we would have found exactly (5.29) as the solution.

along y direction, while maintaining the ratio π_{0x}/π_{0T} fixed. In this case, all terms with derivative on n_{0x} die away, so we have

$$\partial_+\partial_- a_x^{(2)} = -\frac{1}{2n_{0x}}\tilde{f} g^{(1,1)} - \frac{1}{2n_{0x}}\tilde{g} g^{(2,0)}\,, \tag{5.30}$$

$$\partial_+\partial_- b_x^{(2)} = -\frac{1}{2n_{0x}} f \tilde{g}^{(1,1)} - \frac{1}{2n_{0x}} g \tilde{g}^{(2,0)}\,, \tag{5.31}$$

$$\partial_+\partial_- a_y^{(2)} = -\frac{1}{2n_{0x}}\tilde{f} f^{(1,1)} - \frac{1}{2n_{0x}}\tilde{g} f^{(2,0)}\,, \tag{5.32}$$

$$\partial_+\partial_- b_y^{(2)} = -\frac{1}{2n_{0x}} f \tilde{f}^{(1,1)} - \frac{1}{2n_{0x}} g \tilde{f}^{(2,0)}\,. \tag{5.33}$$

Since we saw the effect of $u(y)$ is an overall drift of the system along y direction, we could safely turn it off in the first order perturbation, f, g, $\tilde{f}$, and $\tilde{g}$. Then the first two are oscillatory function of x^+ while the latter two are oscillatory functions of x^-. All terms on the right hand side are of the form,

$$h(x^+)\times\ \tilde{h}(x^-)\,, \tag{5.34}$$

for a pair of some oscillatory functions h and $\tilde{h}$. Such combination of force terms on the right hand side cannot generate a net effect, when averaged over time, since the resonance effect cannot occur. The right hand sides will drive some oscillation of a and b and do not lead to any instability.

Thus, we conclude that this large class of classical solutions are all stable under perturbation and any possible fluctuations move freely along a direction set by electric flux lines. The origin of this one-dimensional behavior was previously explained in terms of collapse of an effective-causal-structure. We again emphasize that the integrability condition can at most restrict acceptable solution to the fluid equations, so stability under the latter is sufficient to argue the stability under the full field equation.

6. Hamiltonian dynamics with potential V

Much of what we studied above concern a region where tachyon rolls to one side of potential. For example we are imagining that $T\to\infty$ everywhere. In this section, we take into account possibility of $V\neq 0$ in the dynamics. The formulation here should serve useful tool for understanding initial stage of tachyon condensation.

6.1 Canonical field equations with V

Let us consider the dynamics with $V\neq 0$ somewhere. For simplicity, we will consider unstable D2 brane case again, with all transverse scalars suppressed. Recall that we are using the notation introduced in section 2, where a fictitious direction x^T is employed and T is treated as if it is a component of the gauge field along x^T. For more details we refers the reader to eqs. (4.14), (4.15).

Half of the hamiltonian equations of motion

$$\vec{E} = \frac{1}{H}(\vec{\pi} - \vec{B}\times\vec{P})\,, \tag{6.1}$$

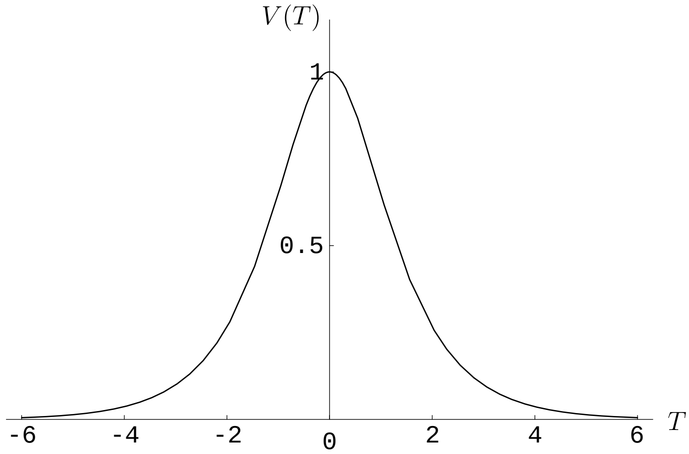

Figure 1: A prototypical form of the potential $V(T)$ as function of T, $V(T) = 1/\cosh T$ [16, 37, 38, 39, 40]. We will use this potential for plot of the time evolution of flux later in this section.

generates the evolution equation for $\vec{B} = (\partial_y T, -\partial_x T, B)$ when combined with the Bianchi identity

$$\dot{\vec{B}} = \vec{\partial} \times \vec{E}\,. \tag{6.2}$$

It is important to note that the energy density H has a V^2 term inside the square root,

$$H = \sqrt{\vec{\pi}^2 + \vec{P}^2 + V^2(1 + \vec{B}^2)}\,. \tag{6.3}$$

The other half gives evolution equations for electric flux $\pi_{x,y}$ and conjugate momentum for tachyon π_T,

$$\dot{\pi}_i + \partial_j \left(\frac{\pi_i P_j - \pi_j P_i + V^2 F_{ij}}{H} \right) = 0\,, \tag{6.4}$$

$$\dot{\pi}_T + \partial_j \left(\frac{\pi_T P_j - \pi_j P_T - V^2 \partial_j T}{H} \right) = -\frac{VV'(1 + \vec{B}^2)}{H} \tag{6.5}$$

with $V' \equiv \partial V/\partial T$ and $F_{ij} = B\epsilon_{ij}$.

These evolution equations should be consistent with energy-momentum conservation which now takes the modified form

$$\dot{H} + \partial_i P_i = 0\,, \tag{6.6}$$

$$\dot{P}_i + \partial_j \left(\frac{P^i P^j - \pi^i \pi^j - V^2(\delta^{ij} + B^i B^j)}{H} \right) = 0\,, \tag{6.7}$$

which is almost identical to the previous case except for V^2 terms. We use the following facts which are the flat metric version of eq. (4.7),

$$\begin{aligned} T^{00} &= \frac{V}{\sqrt{\tilde{X}}} C_S^{00} = H\,, \\ T^{0i} &= \frac{V}{\sqrt{\tilde{X}}} C_S^{0i} = P^i\,, \\ T^{ik} &= \frac{V}{\sqrt{\tilde{X}}} C_S^{ik} = \frac{P^i P^k - \pi^i \pi^k - V^2(\delta^{ik} + B^i B^k)}{H}\,. \end{aligned}$$

In all of above equations, it is quite clear that the fluid-like behavior of $V = 0$ regime will be ruined by the potential, and its effect is of order V^2.

6.2 Flux motion near domain wall formation

In tachyon condensation, $V \neq 0$ can survive the decay process if there is a topological defect. In case of single unstable Dp branes, only possible topological defect would be domain walls, separating a region of $T = \infty$ from that of $T = -\infty$, and at the end of day become $D(p-1)$ branes in the context of type-II theories. We will concentrate on initial configuration which will lead to a single flat D$(p-1)$ brane. With such initial configuration, we would like to ask how flux behaves where $V \neq 0$.

To answer this question, we consider an initial configuration of T such that it has spatial variation along y direction and vanishes at $y = 0$. In addition we assume static initial condition, so that neither the tachyon matter nor the string fluid has initial velocity. Finally for the sake of simplicity, we further assume a uniform distribution of flux lines, lined up along x direction. This can be summarized by the following initial conditions,

$$T_0 = T(y)\,, \qquad \vec{\pi}_0 = (\pi_{0x}, 0, 0)\,, \qquad B = 0\,, \tag{6.8}$$

where π_{0x} is a constant.

Now let us consider evolution of π_i, $(i = x, y)$ as follows,

$$\pi_i(t) = \pi_{i0} + t\dot{\pi}_i|_{t=0} + \frac{1}{2}t^2\ddot{\pi}_i|_{t=0} + \dots\,. \tag{6.9}$$

Under the initial condition (6.8), the first non-trivial variation of the flux appears in $\ddot{\pi}_i|_{t=0}$ term in eq. (6.9), i.e., $\dot{\pi}_i|_{t=0} = 0$, and is given by[7]

$$\begin{aligned} \ddot{\pi}_x &= -\partial_y\left(\frac{\pi_x\dot{P}_y + V^2\dot{B}}{H}\right)_{t=0} \\ &= \left[\frac{V^2}{H}\partial_y^2\left(\frac{\pi_x}{H}\right) - \frac{\pi_x}{H}\partial_y^2\left(\frac{V^2}{H}\right)\right]_{t=0}\,, \end{aligned} \tag{6.11}$$

while π_y is not generated up to this order,

$$\dot{\pi}_y = 0\,, \qquad \ddot{\pi}_y = 0\,. \tag{6.12}$$

Thus, we find the leading order time-variation of flux lines is summarized as

$$\pi_x = \pi_x^0 + \frac{1}{2}t^2\left[\frac{V^2}{H}\partial_y^2\left(\frac{\pi_x}{H}\right) - \frac{\pi_x}{H}\partial_y^2\left(\frac{V^2}{H}\right)\right]_{t=0} + O(t^3)\,. \tag{6.13}$$

[7]Note that at $t = 0$ the energy density H reduces to

$$\sqrt{\pi_x^2 + V^2(1 + (\partial_y T)^2)}\,,$$

while

$$\dot{B} = -\partial_y\left(\frac{\pi_x}{H}\right)_{t=0}\,, \qquad \dot{P}_x = 0\,, \qquad \dot{P}_y = \partial_y\left(\frac{V^2}{H}\right)_{t=0}\,. \tag{6.10}$$

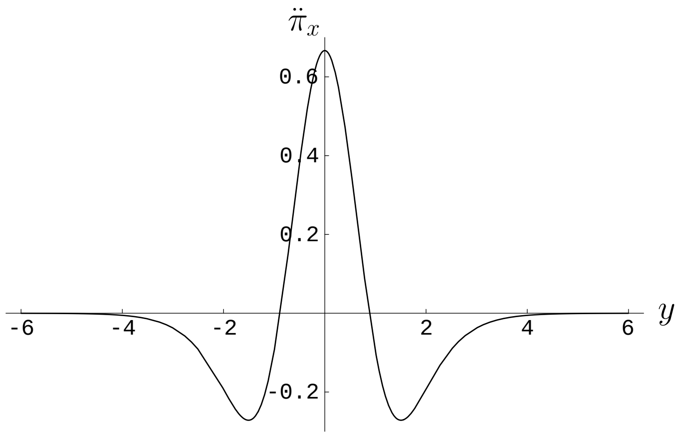

Figure 2: Plot of $\ddot{\pi}_x$ as function of y, with $V(T) = 1/\cosh T$ and $T = y$. It shows short range attraction of fluxes toward $T = 0$ during domain wall formation. At late time, $T = uy$ with $u \to \infty$, so the range of the attractive force infinitesimal. The final configuration at $u = \infty$ corresponds to BPS D$(p-1)$ brane with some fundamental string flux trapped.

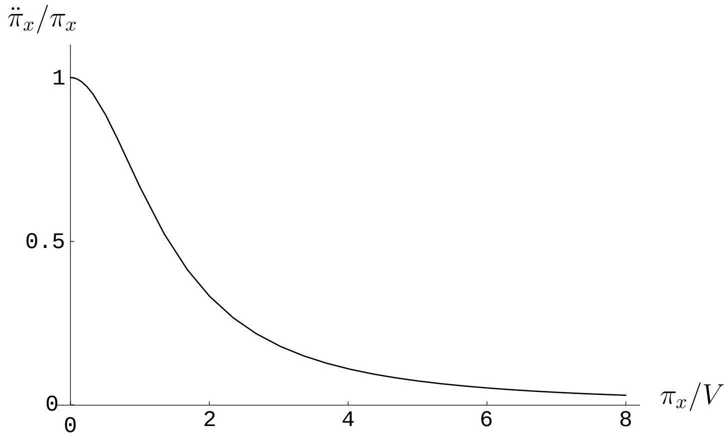

Figure 3: Plot of $\ddot{\pi}_x/\pi_x$ at $y = 0$ as a function of π_x/V at $y = 0$, again at $t = 0$ with the initial condition $T = y$.

From this expression, it is not too difficult to show that initially uniform π_x will get redistributed such that flux in the region $V \sim 1$ tends to gather toward $T = 0$. Since this effect is of order V^2, which is exponentially small far away from $T = 0$ domain walls, the interaction is of short range. The effective range of this attraction is determined by values of V and therefore by the gradient of T along y. With $T = uy$ and in unit where $\alpha' \sim 1$, the range is roughly $\delta y \sim 1/u$. Thus, the effect is maximal when the tachyon begins to roll and die away quickly as the tachyon condensation progresses; To reach final stationary state, u has to be infinite, as is well known.

To illustrate this effect graphically, we draw two figures. Figure 2 is the plot of $\ddot{\pi}_x$ as function of y, assuming an initially linear gradient of $T = y$. This clearly shows local attraction of fluxes toward $T = 0$. Figure 3 is a plot of $\ddot{\pi}_x/\pi_x$ at $y = 0$ as a function of π_x/V at $y = 0$, which shows that the attraction is universal and tends to get weakened when flux energy dominates over the tachyon energy. This effect explains the phenomenon observed in ref. [36] and produces D$(p-1)$ branes and fundamental strings.

6.3 Confined flux strings are exceptional and rare final state

Some of more recent proposals for fundamental string formation utilizes configurations involving $T = 0$, which would be center of domain wall separating regions of $T \to \infty$ and of $T \to -\infty$ [41, 36, 42]. In the above, we analyzed the classical dynamics in such situations, and gave a quantitative description on aggregation of flux lines near domain walls with $T = 0$ at the center. We should note here that any such process will involve bound states of fundamental string with $D(p-1)$ branes as the final product, and the fundamental string flux will be confined to a co-dimension-one hyper-surface rather than a thin, string-like, flux bundle.

We must emphasize here that the real question lies not in whether there are confined flux string but rather how such confined configuration become generic configuration. In fact, classical solution (even with $V = 0$ everywhere) that behaves exactly like a Nambu-Goto string has been introduced in ref. [20] and further studied in ref. [43]. This classical solution admits infinitely many marginal deformation that disperse the flux lines, which makes it very unlikely configuration to form, to begin with.

Decay processes involving topological defects are intriguing in that it could lead to this Nambu-Goto string by fine-tuning initial configurations. For instance, one could imagine a infinitely long cylinder, $R \times S^{p-2}$, of a domain wall to which nearby electric flux get attached. Depending one details of the initial condition, it is possible to imagine that S^{p-2} part of the cylindrical domain wall collapses after tachyon condensation completes and perhaps push all electric flux on its world-volume to an infinitesimally thin string. Such configuration will also obey a Nambu-Goto dynamics classically [44]. However, the trouble is simply that, in the phase space of this theory, such string-like configurations are very exceptional ones. One must still understand exactly how this huge degeneracy is lifted, be it from higher derivative correction or via some quantum effect [8, 9, 45].

7. BPS limit and strings orthogonal to D$(p-1)$ branes

Fundamental strings orthogonal to a D$(p-1)$ brane preserves 1/2 of supersymmetry left unbroken by the D-brane, so the system of such fundamental strings and a D-brane should preserves 1/4 of 32 supersymmetry in spacetime. With the domain wall incorporated into the system, we could ask whether there is such a BPS-like solution involving string fluid transverse to D$(p-1)$ branes. Generic solutions with right spacetime charges was written down in ref. [31].

This smoothness of the domain wall solution may be contrasted to the singular solutions considered in ref. [4]. This smoothness of the solution caution us against in using singular domain walls in the presence of transverse string fluid. It was recently argued that string fluid would be repelled by D$(p-1)$ brane, if the former try to impinge upon the latter [44]. This is based on energetics of the string fluid assuming a singular domain wall background. The smoothness of the solution above is generically attributable to the presence of transverse string fluid, which shows that backreaction due to string fluid may be important in some cases.

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In this section, we will rediscover these smooth solution via our formulation. In particular, we find that a minimal solution of a sort exists and saturate the right BPS energy bound, expected of fundamental strings orthogonal to a D$(p-1)$ brane.

7.1 Smooth domain walls threaded by string fluid

When V is included in the analysis, we may consider a different kind of static solutions, involving domain walls. A singular solution of domain wall type has been known [4], but recently it was found that a smooth domain wall solution is possible when we modify the asymptotic boundary condition [31, 32, 33]. Here we will reproduce these solutions within our formalism and study its properties.

For this, we will consider static configurations with $P_i = 0$ and $\pi_T = 0$, and assume a uniform electric flux, π_x. We will achieve $P_i = 0$ by setting $F_{ij} = 0$. With this, another allowed quantity is

$$v_T = n_x \partial_x T \,. \tag{7.1}$$

In such configurations, a useful identity is

$$H = \sqrt{(\pi_x^2 + V(T)^2)(1 + (\partial_x T)^2)}\,. \tag{7.2}$$

Then the momentum conservation (6.7) forces

$$0 = \partial_x \left(\frac{\pi_x^2 + V(T)^2}{H} \right) = \partial_x \left(\frac{\pi_x^2 + V(T)^2}{1 + (\partial_x T)^2} \right)^{1/2}, \tag{7.3}$$

which may be rewritten as

$$C^2 \left(\pi_x^2 + V(T)^2 \right) = 1 + (\partial_x T)^2 \tag{7.4}$$

for some constant C. Inspection of (6.1) shows that this constant is related to the asymptotic value of $E_x \to \epsilon$ as $(C\pi_x)^2 = \epsilon^2$.[8] Thus we only need to solve

$$\frac{dT}{dx} = \pm \sqrt{\epsilon^2 \left(1 + \frac{V(T)^2}{\pi_x^2} \right) - 1}\,. \tag{7.5}$$

There are three generic cases to consider

- If ϵ^2 is larger than or equal to 1, this gives monotonic solution $T(x)$, and represents a single D$(p-1)$ (or anti-D$(p-1)$) brane at $T = 0$.
- If ϵ^2 is less than 1 but larger than $1/(1 + V(0)^2/\pi_x^2)$, this gives an oscillatory $T(x)$ with an infinite array of zeros and represents an alternating array of D$(p-1)$ and anti-D$(p-1)$ branes.
- If ϵ^2 is less than $1/(1 + V(0)^2/\pi_x^2)$, no solution exists.

In all cases where solutions exist, the configuration is that of smooth domain wall solutions, generically threaded by uniform distribution of flux lines transverse to the domain wall.

[8] For the present solutions E_x itself is a constant, by the way.

7.2 A minimal solution and BPS energy

Energy density of the above solutions are

$$H = \sqrt{(\pi_x^2 + V^2)(1 + (\partial_x T)^2)} = \frac{\epsilon}{|\pi_x|}(\pi_x^2 + V^2) \tag{7.6}$$

which approaches $\epsilon\pi_x$ asymptotically. We are interested in solution with pure string fluid far away from domain wall, and this forces us to consider $\epsilon = 1$, in particular. This choice is also energetically favored since it represents the lowest energy solution available, given the conserved flux and the single domain wall. As we will see shortly, this case may be regarded as a BPS-saturated solution.

As would be expected from $E_x^2 = 1$, the equation of motion simplifies to

$$\frac{dT}{dx} = \pm\frac{V(T)}{\pi_x} \tag{7.7}$$

whose solution is such that T does approach vacuum at $x = \pm\infty$. The energy density also simplifies further for $\epsilon = 1$ as,[9]

$$H = |\pi_x| + \frac{V^2}{|\pi_x|} = \pi_x + V|\partial_x T|\,. \tag{7.8}$$

The first piece is the BPS energy density associated with the fundamental string charge. The second piece must be associated with the domain wall itself. In fact, integrating over x, this is precisely the tension of the D$(p-1)$ brane

$$\int V\,\partial_x T\,dx = \int_{-\infty}^{\infty} V\,dT = \tau_{p-1}\,. \tag{7.9}$$

The energy of the solution is then precisely sum of two terms, one from tension of the D$(p-1)$ brane and the other from transverse fundamental string. This gives exactly the BPS energy expected of fundamental string ending on or passing through a D-brane.

Thus at least when we have uniform string fluid threading a D$(p-1)$-brane, the string fluid can mimic, quantitatively, behavior of fundamental string ending on D-branes. This may be compared with findings of ref. [44], where it is asserted that string fluid cannot end on D$(p-1)$ brane. The latter is based on a computation where an infinitely thin D$(p-1)$ is taken to be a background and neglects a possible backreaction of the domain wall solution to the presence of string fluid. While the assertion might stand when transverse string flux are well-isolated, this example cautions us to be careful about backreaction of a domain wall.

8. Summary

We have reviewed dynamics of the tachyon coupled to a gauge field. When the tachyon condensation has progressed far so that $V \to 0$, the dynamics is that of two fluids, string

[9] We are indebted to Chanju Kim on this point.

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fluid and tachyon matter, as anticipated, but the fluid equation of motion must be augmented by a set of integrability condition, which is necessary if we wish to recover A_μ and T from the fluid variables. We have isolated a large family of static solutions with huge degeneracy and tested their stability.

We further extended the formalism to the case of $V \neq 0$ somewhere, and showed that during initial stage of condensation, electric flux lines tends to be attracted toward $T = 0$. Since $T = 0$ may survive the condensation process only if there is a domain wall, this initial configuration will lead to a D$(p-1)$ brane with fundamental string flux spread on it. Interaction between a domain wall and transverse string fluid is more drastic, and the latter thickens the former. We also wrote down such smooth domain wall solution, using the hamiltonian formulation, and discovered that minimal solution of this type saturates a BPS bound of strings ending on D-branes.

One aspect of the $V = 0$ limit, worth emphasizing, concerns the coupling between the two fluid components. Despite a rather tight coupling between the string fluid and the tachyon matter, as evidenced by the form of the hamiltonian, static distribution of the two fluid components seems pretty much independent of each other. The only real constraint is that static distribution of tachyon matter must be uniform along the string fluid direction. One might have expected that two fluid components come with the same (transverse) distribution, but this is not the case at all. Instead, the tight coupling between the tachyon matter and the string fluid affects the subsequent dynamics in an unexpected way. As is clear from the fluid equation, distribution of string fluid and distribution of tachyon matter evolves in the complete absence of pressure (except for the tension along the string fluid) and is essentially free. That is they simply responds to velocity field of fluid, which is in turn affected by gradient of density of flux direction along the flux lines. Regardless of the details of the latter, however, two fluid shares one and the same velocity field v. This implies that once the relative distribution of the two fluid component is set, the subsequent perturbation moves string fluid and tachyon together in such a manner that keeps this ratio fixed when followed by co-moving observers.

The classical system of tachyon and gauge field turned out to have many intriguing surprises; fluid-like behavior of the perfectly sensible field theory, the drastic reduction of perturbative degrees of freedom, Carollian collapse of effective light-cone, and finally many intriguing classical solutions, static or dynamic, singular or nonsingular. One question that remains murky is whether this classical field theory admits a sensible and practical quantum treatment of its own, regardless of its tie to stringy context, and whether interesting physics arise from a quantization. We feel that this avenue of research deserves more study in future.

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A. S-dual formulation of $V = 0$ limit

Ref. [20] discovered a nonvanishing dual lagrangian for pure gauge system, whose equation of motions and Bianchi identity are respectively Bianchi identity and equation motion of the original Born-Infeld system. It is rather clear that dualization procedure works even when scalar fields are present, once we think of the latter as components of the gauge field along some fictitious dimensions.

The prescription for the dualization process is to introduce a new gauge field C and its field strength, $G = dC$ such that

$$\mathcal{K} \equiv \pi_m \, dt \wedge dx^m + \frac{1}{2} K_{mn} dx^m \wedge dx^n = *G \,. \tag{A.1}$$

Momentum π_m is canonical conjugate momenta of $A_m = (A_1, \ldots, A_p, T)$. K_{mn} are defined as

$$K_{mn} = \frac{\delta \mathcal{H}}{\delta F_{mn}} \,. \tag{A.2}$$

Finally, the Hodge-dual operator $*$ here is defined with respect to the "volume form"

$$dt \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p \wedge dx^T \,. \tag{A.3}$$

With this, the Legendre transformation of the hamiltonian gives a dual lagrangian as a function of the dual field strength G,

$$\mathcal{L}' = \sqrt{\frac{-\mathcal{K}^2}{2}} = \sqrt{\frac{G^2}{2}} \,. \tag{A.4}$$

One crucial point here is that the Legendre transformation that leads to $\mathcal{L}'$ is not reversible. Thus, to obtain truly dual lagrangian, one must introduce constraints on the dual side.

As explained in ref. [20], this does not complete the dualization process due to the fact that the original hamiltonian is quite degenerate. That is, part of F_{mn} which has no inner product with π_m never enters the hamiltonian. Because of this, inverse Laplace transform of $\mathcal{L}'$ does not give us back $\mathcal{H}$. This problem is curable by adding constraints,

$$\mathcal{K} \wedge \mathcal{K} = 0 \,. \tag{A.5}$$

Thus the correct dual lagrangian may be written as

$$\sqrt{\frac{G^2}{2}} + \langle \lambda, *G \wedge *G \rangle \tag{A.6}$$

with a Lagrange multiplier field λ. This is where the pure gauge dynamics maps to that of ref. [46], which attempted to write a field theory that produces string-like degrees of freedom classically.

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B. Canonical field equation with $V = 0$

The tachyon effective action with gauge field is given by

$$S = \int d^{p+1}x \ \mathcal{L} \,, \tag{B.1}$$

where

$$\begin{aligned} \mathcal{L} &= -V(T)\sqrt{-X} \,, \\ X_{\mu\nu} &\equiv g_{\mu\nu} + F_{\mu\nu} + \partial_\mu T \partial_\nu T \,, \\ X &\equiv \det X_{\mu\nu} \,. \end{aligned} \tag{B.2}$$

Then the determinant X can be expressed as

$$X = X_{00} \det(X_{ij}) - X_{0i} D^{ij} X_{j0} \,, \tag{B.3}$$

where the matrix D is transpose of the cofactor for matrix $(X)_{ij}$ and the components of matrix $(X)_{\mu\nu}$ are written by

$$\begin{aligned} X_{00} &= -N^2 + h_{ij} L^i L^j + \dot{T}^2 \,, \\ X_{0i} &= E_i^+ \,, \\ X_{i0} &= -E_i^- \,, \\ X_{ij} &= h_{ij} + F_{ij} + \partial_i T \partial_j T \end{aligned}$$

with

$$E_i^\pm \equiv F_{0i} \pm \dot{T} \partial_i T \pm L^j h_{ji} \,. \tag{B.4}$$

Let us denote the hamiltonian using canonical variables to describe the dynamics of the system by hamiltonian equations. Then conjugate momenta are given by

$$\pi^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{V}{\sqrt{-X}} \frac{E_j^+ D^{ji} + D^{ij} E_j^-}{2} \,, \tag{B.5}$$

$$\pi_T \equiv \frac{\partial \mathcal{L}}{\partial \dot{T}} = \frac{V}{\sqrt{-X}} \left(\dot{T} \det X_{ij} - \frac{E_j^+ D^{ji} \partial_i T - \partial_i T D^{ij} E_j^-}{2} \right) , \tag{B.6}$$

where π^i (π_T) is conjugate to A_i (T), and π^i satisfies the Gauss constraint $\partial_i \pi^i = 0$. The hamiltonian is obtained by the following Legendre transformation

$$\mathcal{H} = \pi^i E_i + \pi_T \dot{T} - \mathcal{L} + \pi^i \partial_i A_0 \,. \tag{B.7}$$

From now on, let us use matrix notation for simplicity. Then the quantities which we have defined are denoted by in matrix forms in temporal gauge $A_0 = 0$,

$$X = X_{00} \det X_{ij} + E^+ D E^- \,, \tag{B.8}$$

$$\pi = \frac{V}{\sqrt{-X}} \frac{E^+ D + D E^-}{2} \,, \tag{B.9}$$

$$\begin{aligned}
\pi_T &= \frac{V}{\sqrt{-X}}\left(\dot{T}\det X_{ij} - \frac{E^+D\partial T - \partial T D E^-}{2}\right), && \text{(B.10)}\\
\mathcal{H} &= \frac{V}{\sqrt{-X}}\left(\frac{E^+DE + EDE^-}{2} + \dot{T}^2\det X_{ij} - \frac{E^+D\partial T\dot{T} - \dot{T}\partial T D E^-}{2} - X\right)\\
&= \frac{V}{\sqrt{-X}}\left((N^2 - LhL)\det X_{ij} + \frac{E^+DhL - LhDE^-}{2}\right), && \text{(B.11)}
\end{aligned}$$

where all matrix indices are $i,j,k = 1,\ldots,p$. Let us define a matrix for convenience in calculations

$$\bar{X} = h + F + \partial T\partial T\,, \tag{B.12}$$

which has properties

$$D\bar{X} = \bar{X}D = \det X_{ij} I\,, \tag{B.13}$$

where I is $p\times p$ unit matrix. Then we obtain the following relations

$$\begin{aligned}
\pi\bar{X}\pi &= \pi h\pi + (\pi\partial T)^2\,, && \text{(B.14)}\\
F\pi + \partial T\pi_T &= \frac{V}{\sqrt{-X}}\left(\dot{T}\partial T\det X_{ij} - \frac{E^+ - E^-}{2}\det X_{ij} + \frac{E^+Dh - hDE^-}{2}\right).
\end{aligned}$$

Using the relations in eq. (B.14), we find

$$\sqrt{\pi h\pi + \pi_T^2 + (F\pi + \partial T\pi_T)h^{-1}(F\pi + \partial T\pi_T) + (\pi\partial T)^2 + V^2\det X_{ij}} = \frac{V}{\sqrt{-X}}N\det X_{ij}\,, \tag{B.15}$$

where the matrix h^{-1} has components $(h^{-1})_{ij} = h^{ij}$. Now we can rewrite the hamiltonian representation using canonical variables as

$$\begin{aligned}
\mathcal{H} &= \frac{V}{\sqrt{-X}}N^2\det X_{ij} - \pi FL + \pi_T\partial L\\
&= N\sqrt{\pi h\pi + \pi_T^2 + (F\pi + \partial T\pi_T)h^{-1}(F\pi + \partial T\pi_T) + (\pi\partial T)^2 + V^2\det X_{ij}} -\\
&\quad -\pi FL + \pi_T\partial TL\,, && \text{(B.16)}
\end{aligned}$$

where we used the relation

$$E^+DLh - LhDE^- = 2\left(LhL\det X_{ij} - \frac{\sqrt{-X}}{V}\pi FL + \frac{\sqrt{-X}}{V}\pi_T\partial TL\right). \tag{B.17}$$

Then the hamiltonian equations in $V=0$ limit are given by

$$\begin{aligned}
\dot{T} &= \frac{\partial\mathcal{H}}{\partial\pi_T} = \frac{N}{\sqrt{Y}}\left(\pi_T(1 + \partial Th^{-1}\partial T) + \partial Th^{-1}F\pi\right) + \partial TL\,, && \text{(B.18)}\\
\dot{\pi}_T &= -\frac{\partial\mathcal{H}}{\partial T} = \partial_i\left[\frac{N}{\sqrt{Y}}\left(\pi^i(\pi\partial T) + \pi_T h^{ij}(F_{jk}\pi^k + \partial_j T\pi_T)\right)\right] + \partial_i(\pi_T L^i)\,, && \text{(B.19)}\\
\dot{A}_i &= \frac{\partial\mathcal{H}}{\partial\pi^i} = \frac{N}{\sqrt{Y}}\left(h\pi + \partial T(\pi\partial T) - Fh^{-1}(F\pi + \partial T\pi_T)\right)_i - (FL)_i\,, && \text{(B.20)}\\
\dot{\pi}^i &= -\frac{\partial\mathcal{H}}{\partial A_i} = \partial_j\left(\frac{N}{\sqrt{Y}}(\pi^i h^{jk} - \pi^j h^{ik})(F\pi + \partial T\pi_T)_k\right) - \partial_j(\pi^j L^i - \pi^i L^j)\,, && \text{(B.21)}
\end{aligned}$$

where $Y \equiv \pi h\pi + \pi_T^2 + (F\pi + \partial T\pi_T)h^{-1}(F\pi + \partial T\pi_T) + (\pi\partial T)^2$.

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C. Second order perturbation equations

General perturbation equations for eqs. (5.1), (5.3) are given by,

$$\partial_- \vec{a}^{(n)} = -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} b_y^{(n)} - \frac{1}{2n_{0x}} \sum_{i=1}^{n-1} (\vec{b}^{(n-i)} \cdot \vec{\partial}) \vec{a}^{(i)} , \tag{C.1}$$

$$\partial_+ \vec{b}^{(n)} = -\frac{\partial_y \vec{n}^{(0)}}{2n_{0x}} a_y^{(n)} - \frac{1}{2n_{0x}} \sum_{i=1}^{n-1} (\vec{a}^{(n-i)} \cdot \vec{\partial}) \vec{b}^{(i)} , \tag{C.2}$$

$$2\vec{n}^{(0)} \cdot \vec{a}^{(n)} = -\sum_{i=1}^{n-1} \vec{a}^{(i)} \cdot \vec{a}^{(n-i)} , \tag{C.3}$$

$$2\vec{n}^{(0)} \cdot \vec{b}^{(n)} = -\sum_{i=1}^{n-1} \vec{b}^{(i)} \cdot \vec{b}^{(n-i)} . \tag{C.4}$$

Now let us consider the second order perturbation equations. The y-components of the second order perturbation equations in eqs. (C.1), (C.2) are written by

$$\begin{aligned} \partial_- a_y^{(2)} &= -\frac{1}{2n_{0x}} (\vec{b}^{(1)} \cdot \vec{\partial}) a_y^{(1)} \\ &= \left(-\frac{1}{2n_{0x}} \tilde{g} + \frac{n'_{0x}}{(2n_{0x})^2} F \right) f^{(1,0)} - \frac{n'_{0x}}{2n_{0x}} t \tilde{f} f^{(1,0)} - \frac{1}{2n_{0x}} \tilde{f} f^{(0,1)} , \end{aligned} \tag{C.5}$$

$$\begin{aligned} \partial_+ b_y^{(2)} &= -\frac{1}{2n_{0x}} (\vec{a}^{(1)} \cdot \vec{\partial}) b_y^{(1)} \\ &= \left(-\frac{1}{2n_{0x}} g + \frac{n'_{0x}}{(2n_{0x})^2} \tilde{F} \right) \tilde{f}^{(1,0)} + \frac{n'_{0x}}{2n_{0x}} t f \tilde{f}^{(1,0)} - \frac{1}{2n_{0x}} f \tilde{f}^{(0,1)} , \end{aligned} \tag{C.6}$$

where we used the results of the first order perturbation (5.8) and (5.10). Then we obtain the solutions of these inhomogeneous first order differential equations,

$$a_y^{(2)} = h(x^+, y) - \left(\frac{1}{2n_{0x}} \tilde{G} + \frac{n'_{0x}}{2n_{0x}} t \left(F + \tilde{F} \right) + \frac{n'_{0x}}{(2n_{0x})^2} \tilde{\mathcal{F}} \right) f^{(1,0)} - \frac{1}{2n_{0x}} \tilde{F} f^{(0,1)} , \tag{C.7}$$

$$b_y^{(2)} = \tilde{h}(x^-, y) - \left(\frac{1}{2n_{0x}} G - \frac{n'_{0x}}{2n_{0x}} t \left(F + \tilde{F} \right) + \frac{n'_{0x}}{(2n_{0x})^2} \mathcal{F} \right) \tilde{f}^{(1,0)} - \frac{1}{2n_{0x}} F \tilde{f}^{(0,1)} , \tag{C.8}$$

where h and $\tilde{h}$ are arbitrary functions and we define,

$$\begin{aligned} G(x^+, y) &\equiv \int^{x^+} dw \; g(w, y) , \qquad & \tilde{G}(x^-, y) &\equiv \int^{x^-} dw \; \tilde{g}(w, y) , \\ \mathcal{F}(x^+, y) &\equiv \int^{x^+} dw \; F(w, y) , \qquad & \tilde{\mathcal{F}}(x^-, y) &\equiv \int^{x^-} dw \; \tilde{F}(w, y) . \end{aligned} \tag{C.9}$$

Using the eqs. (5.8), (5.10), (C.7), we can obtain the second order perturbation equations for the x-components of the eqs. (C.1), (C.2) as follows:

$$\begin{aligned} \partial_- a_x^{(2)} &= -\frac{1}{2n_{0x}} \left(b_y^{(2)} n'_{0x} + (\vec{b}^{(1)} \cdot \vec{\partial}) a_x^{(1)} \right) - \\ &= -\frac{n'_{0x}}{2n_{0x}} \tilde{h} + \frac{n'_{0x}}{(2n_{0x})^2} F \tilde{f}^{(0,1)} + \left(-\frac{1}{2n_{0x}} \tilde{g} + \frac{\partial_y n_{0x}}{(2n_{0x})^2} F - \frac{n'_{0x}}{2n_{0x}} t \tilde{f} \right) g^{(1,0)} + \end{aligned}$$

$$+\left(\frac{n'_{0x}}{(2n_{0x})^2}G+\frac{(n'_{0x})^2}{(2n_{0x})^3}\mathcal{F}-\frac{(n'_{0x})^2}{(2n_{0x})^2}t\left(F+\tilde{F}\right)\right)\tilde{f}^{(1,0)}+$$
$$+\left(\frac{n'_{0x}}{(2n_{0x})^2}\tilde{g}-\frac{(n'_{0x})^2}{(2n_{0x})^3}F-\frac{1}{2n_{0x}}g^{(0,1)}+\frac{n'_{0x}}{(2n_{0x})^2}\tilde{F}^{(0,1)}+\right.$$
$$\left.+\frac{2\left(n_{0x}(\partial_y^2 n_{0x})-(n'_{0x})^2\right)}{(2n_{0x})^3}\tilde{F}-\frac{(n'_{0x})^2}{(2n_{0x})^2}t\tilde{f}\right)\tilde{f}\,, \tag{C.10}$$

$$\begin{aligned}\partial_+ b_x^{(2)} &= -\frac{1}{2n_{0x}}\left(a_y^{(2)}n'_{0x}+(\vec{a}^{(1)}\cdot\vec{\partial})b_x^{(1)}\right)\\ &= -\frac{n'_{0x}}{2n_{0x}}h+\frac{n'_{0x}}{(2n_{0x})^2}\tilde{F}f^{(0,1)}+\left(-\frac{1}{2n_{0x}}g+\frac{\partial_y n_{0x}}{(2n_{0x})^2}\tilde{F}+\frac{n'_{0x}}{2n_{0x}}tf\right)\tilde{g}^{(1,0)}+\\ &\quad+\left(\frac{n'_{0x}}{(2n_{0x})^2}\tilde{G}+\frac{(n'_{0x})^2}{(2n_{0x})^3}\tilde{\mathcal{F}}+\frac{(n'_{0x})^2}{(2n_{0x})^2}t\left(F+\tilde{F}\right)\right)f^{(1,0)}+\\ &\quad+\left(\frac{n'_{0x}}{(2n_{0x})^2}g-\frac{(n'_{0x})^2}{(2n_{0x})^3}\tilde{F}-\frac{1}{2n_{0x}}\tilde{g}^{(0,1)}+\frac{n'_{0x}}{(2n_{0x})^2}F^{(0,1)}+\right.\\ &\quad\left.+\frac{2\left(n_{0x}(\partial_y^2 n_{0x})-(n'_{0x})^2\right)}{(2n_{0x})^3}F+\frac{(n'_{0x})^2}{(2n_{0x})^2}tf\right)f\,.\end{aligned} \tag{C.11}$$

Using these results, we find

$$\begin{aligned}\partial_+\partial_- a_x^{(2)} &= \left(\frac{n'_{0x}}{(2n_{0x})^2}g-\frac{(n'_{0x})^2}{(2n_{0x})^3}\tilde{F}-\frac{(n'_{0x})^2}{(2n_{0x})^2}tf\right)\tilde{f}^{(1,0)}+\\ &\quad+\frac{n'_{0x}}{(2n_{0x})^2}f\tilde{f}^{(0,1)}+\frac{n'_{0x}}{(2n_{0x})^2}(f-\tilde{f})g^{(1,0)}-\frac{1}{2n_{0x}}\tilde{f}g^{(1,1)}+\\ &\quad+\left(-\frac{1}{2n_{0x}}\tilde{g}+\frac{n'_{0x}}{(2n_{0x})^2}F-\frac{n'_{0x}}{2n_{0x}}t\tilde{f}\right)g^{(2,0)}-\frac{(n'_{0x})^2}{(2n_{0x})^3}(f+\tilde{f})\tilde{f}\,,\end{aligned} \tag{C.12}$$

$$\begin{aligned}\partial_+\partial_- b_x^{(2)} &= \left(\frac{n'_{0x}}{(2n_{0x})^2}\tilde{g}-\frac{(n'_{0x})^2}{(2n_{0x})^3}F+\frac{(n'_{0x})^2}{(2n_{0x})^2}t\tilde{f}\right)f^{(1,0)}+\\ &\quad+\frac{n'_{0x}}{(2n_{0x})^2}\tilde{f}f^{(0,1)}-\frac{n'_{0x}}{(2n_{0x})^2}(f-\tilde{f})\tilde{g}^{(1,0)}-\frac{1}{2n_{0x}}f\tilde{g}^{(1,1)}+\\ &\quad+\left(-\frac{1}{2n_{0x}}g+\frac{n'_{0x}}{(2n_{0x})^2}\tilde{F}+\frac{n'_{0x}}{2n_{0x}}tf\right)\tilde{g}^{(2,0)}-\frac{(n'_{0x})^2}{(2n_{0x})^3}(f+\tilde{f})f\,,\end{aligned} \tag{C.13}$$

$$\begin{aligned}\partial_+\partial_- a_y^{(2)} &= \frac{n'_{0x}}{(2n_{0x})^2}(f-\tilde{f})f^{(1,0)}-\frac{1}{2n_{0x}}\tilde{f}f^{(1,1)}-\\ &\quad-\left(\frac{1}{2n_{0x}}\tilde{g}+\frac{n'_{0x}}{2n_{0x}}t\tilde{f}-\frac{n'_{0x}}{(2n_{0x})^2}F\right)f^{(2,0)}\,,\end{aligned} \tag{C.14}$$

$$\begin{aligned}\partial_+\partial_- b_y^{(2)} &= \frac{n'_{0x}}{(2n_{0x})^2}(\tilde{f}-f)\tilde{f}^{(1,0)}-\frac{1}{2n_{0x}}f\tilde{f}^{(1,1)}-\\ &\quad-\left(\frac{1}{2n_{0x}}g-\frac{n'_{0x}}{2n_{0x}}tf-\frac{n'_{0x}}{(2n_{0x})^2}\tilde{F}\right)\tilde{f}^{(2,0)}\,.\end{aligned} \tag{C.15}$$

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